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THE CONSTRUCTION AND USE OF LOGARITHMIC POLE-ZERO, ROOT LOCUS AND FREQUENCY RESPONSE PLOTS

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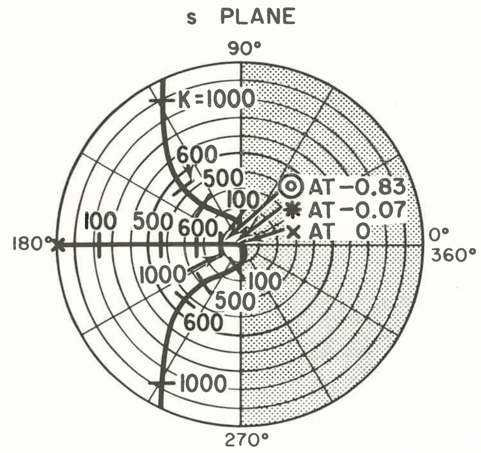
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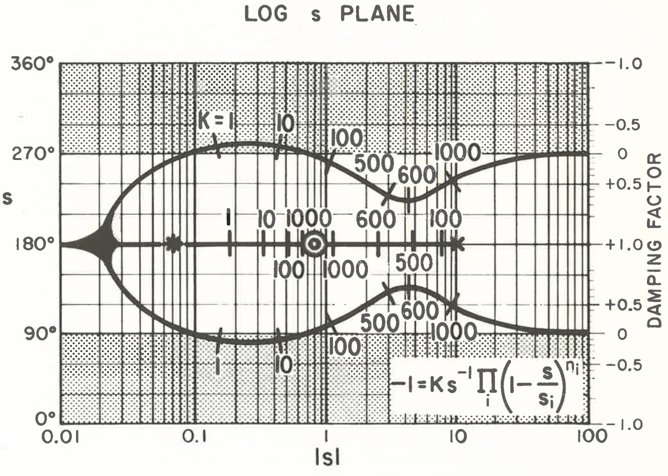
LOG s PLANE

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THE CONSTRUCTION AND USE OF LOGARITHMIC POLE-ZERO, ROOT LOCUS AND FREQUENCY RESPONSE PLOTS

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ABSTRACT

The use of logarithmic coordinates for pole-zero and root locus plots emphasizes their relationship to frequency response plots and encourages the transfer back and forth between the two methods to realize the maximum advantage of each. The combined method will be illustrated and the construction of the plots shown using graphical techniques and using the ESIAC® Algebraic Analog Computer.

INTRODUCTION

As automatic control systems have become more complex through the years, new and more powerful methods of designing and analyzing them have been required. Although studies in the time domain using digital or analog computers provide obvious answers for a system with specified parameters, transform methods are extremely important in the design of a system with a large number of variable parameters. Examples of transform methods are equations involving the complex frequency $j\omega$, the Laplace transform, the Fourier transform, and the z and w transforms used for sampled data systems. Such transform methods replace differential equations by algebraic transfer functions and allow use of a wide variety of performance criteria and aids to visualization, such as Nyquist diagrams, Nichols charts, pole-zero plots, frequency response curves and root locus plots.

Perhaps the most widely used method is the logarithmic frequency response plot, or Bode plot.[1] It is particularly easy to sketch, since it can be constructed, first as an asymptote plot, with straight line segments; then corrected, if necessary, to produce exact gain and phase curves. When a system has more than one lightly-damped resonant response, however, the Bode plot may not provide sufficient information. In this case it is helpful to determine the pole locations of the closed-loop transfer function by the root locus method introduced by Evans.[2] An excellent survey of various methods, with a discussion of their advantages and disadvantages, has been made by Axelby.[3]

Frequency response methods in system design are far older than root locus methods, and their common origin and similarities are often overlooked. Their relationship has been obscured by the fact that, though frequency response plots are commonly drawn in logarithmic coordinates, pole-zero diagrams and root locus plots have been constructed in the past in linear coordinates. A pole-zero diagram gives a graphical picture of all the values of s at which a transfer function becomes equal to infinity or zero. This specifies all the mathematical terms, or modes, in the response of the system to any signal--sinusoidal or transient. When a designer makes a change in a system, he can use the root locus method to construct the new pole-zero diagram from the original one. The most familiar root locus application is the determination of the closed-loop transfer function of a feedback system from its open-loop function.

The use of logarithmic coordinates for pole-zero plots is just as valuable as for frequency response plots. The designer needs to see all the zeros and poles over a wide magnitude range, just as he needs a wide frequency range in frequency response methods. The Bode method of constructing frequency response curves requires the use of logarithmic coordinates; the usual graphical root locus construction--Evans' method--is applicable only in linear coordinates. Such root locus plots suffer from too limited a range--less than one decade can be shown with reasonable detail.

The use of the linear complex plane is a natural one. In the simple s plane, the vector joining any two points represents the difference between their values of s, making possible calculations by graphical construction. However, other calculation methods not based upon vector measurements are available for use in other

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Superior numbers refer to similarly-numbered references at the end of this paper.

coordinate systems. After a little experience with pole-zero plots and root locus diagrams in the log s plane, its advantages over the s plane become evident, both in its ability to cover poles and zeros over a wide frequency range with equal emphasis, and in its direct relationship to logarithmic frequency response plots.

THE BASIC EQUATION

The transform methods mentioned, including frequency response methods, are all variations on the basic Laplace transformation. In the Laplace transform method, the signals in a system and the relationships between them are all represented by functions of a complex variable s. These functions are normally algebraic functions which can be factored to express them in either of two equivalent forms:

$$\begin{aligned} F &= K_1 s^{n_0} (s-s_1)^{n_1} (s-s_2)^{n_2} (s-s_3)^{n_3} \cdots \\ &= K_1 s^{n_0} \prod_i (s-s_i)^{n_i} \end{aligned} \qquad (1)$$

or:

$$\begin{aligned} F &= K_2 s^{n_0} \left(1-\frac{s}{s_1}\right)^{n_1} \left(1-\frac{s}{s_2}\right)^{n_2} \left(1-\frac{s}{s_3}\right)^{n_3} \cdots \\ &= K_2 s^{n_0} \prod_i \left(1-\frac{s}{s_i}\right)^{n_i} \end{aligned} \qquad (2)$$

Note that only the K values differ. The K's in the two equations are related by the equation:

$$K_2 = K_1 \prod_i (-s_i)^{n_i} \qquad (3)$$

In these equations the symbol $\prod$ is used to indicate a product of factors in a manner similar to the use of $\sum$ to indicate a summation of terms. F, s and the s_i's are complex quantities. For rational functions, K_1 or K_2 will be real and the n's will be integers, both positive and negative. The s_i values are called the zeros and poles of the function--zeros in factors with positive exponents and poles in factors with negative exponents. The absolute value of n_i associated with each s_i is called the order of the zero or pole.

The zeros and poles of a function are the values of s at which it is equal to zero or infinity. The roots of an equation in the form of Equation (1) or (2) are the values of s at which F is equal to some specified value other than zero or infinity. In a problem where roots are to be determined, the zeros and poles of the function and the values of F and K are all specified parameters. The difference between a zero or pole and a root should be kept clearly in mind. The fact that the roots of one equation may be the zeros or poles of the function used in the next calculation has sometimes caused confusion.

MAGNITUDE AND ANGLE EQUATIONS

For the practical evaluation of Equation (1) or (2), the logarithm of both sides of the equation may be taken and the real and imaginary parts separated. This gives independent equations for the magnitude and the angle of the function. Using $|\,|$ to denote magnitude, and $\angle$ to represent angle, Equation (1) becomes:

$$\log |F| = \log |K_1| + n_0 \log |s| + \sum_i n_i \log |s-s_i| \qquad (1a)$$

and

$$\angle F = \angle K_1 + n_0 \angle s + \sum_i n_i \angle (s-s_i) \qquad (1b)$$

and Equation (2) becomes:

$$\log |F| = \log |K_2| + n_0 \log |s| + \sum_i n_i \log \left|1-\frac{s}{s_i}\right| \qquad (2a)$$

and

$$\angle F = \angle K_2 + n_0 \angle s + \sum_i n_i \angle \left(1-\frac{s}{s_i}\right) \qquad (2b)$$

In these equations the product of factors has been converted to a sum of terms in which the value of each term is real, not complex. The antilogarithm of the magnitude equation is sometimes taken, to express the magnitude of the product as the product of the magnitudes of the factors. The angle of the product, however, is always the sum of the angles of the factors. The magnitude and angle equations are the basis for analog methods of evaluating Equation (1) or (2), whether the analog is graphical, mechanical or electrical.

The magnitude and angle equations are useful both in problems where values of s are given and values of the function calculated, and in problems where F is given and the roots required. Frequency response calculations are an example of the first type of problem. Here, s is replaced by the angular frequency $j\omega$, and $|F|$ and $\angle F$ are the gain and phase shift respectively. Although the calculation of gain and phase at any frequency is straightforward, the construction of frequency response curves would be very time-consuming were it not for the Bode method.[1]

An example of the second type of problem is the root locus method of control system analysis. Here the zeros and poles of an open-loop transfer function are used in an equation whose roots are the poles of the closed-loop response. Finding the roots would be like looking for the proverbial needle in a haystack, were it not for the fortunate fact that to each needle, a thread is attached. In a complex plane plot, each root must lie along a line, or locus, which is a solution of the angle equation, without regard for the magnitude equation. These loci are easily found, since they begin and end at the zeros and poles of the function. The root locus plotting method of Evans[2] provided the necessary speed-up to make the method practical.

Both the Bode method and the Evans method make use of the logarithmic magnitude and angle equations. Bode plots also make use of logarithmic coordinates, but root locus plots in Evans' graphical construction could not make use of a logarithmic complex plane. Furthermore, the linear complex plane is taught to every elementary student of complex variables, but the logarithmic plane is seldom considered. As a result, the advantages of logarithmic coordinates for pole-zero diagrams and root locus plots have not been generally recognized.

THE s PLANE VERSUS THE LOG s PLANE

The zeros and poles of a function can be plotted as points in any two-dimensional coordinate system for the complex variable. It is customary to use a circle to plot a zero and an x to mark a pole. A plot of all the zeros and poles of a function, with notes indicating the order of any higher order zeros and poles, completely specifies the function except for the value of K_1 or K_2.

Various coordinate systems can be used for such plots. The most common one for the variable s is called the s plane. Its rectangular (Cartesian) coordinates are the real and imaginary components of s--$\text{Re}[s]$ and $\text{Im}[s]$. Its polar coordinates are the magnitude and angle of s--$|s|$ and $\angle s$. These alternate coordinates for the s plane are shown in the left half of Figure 1.

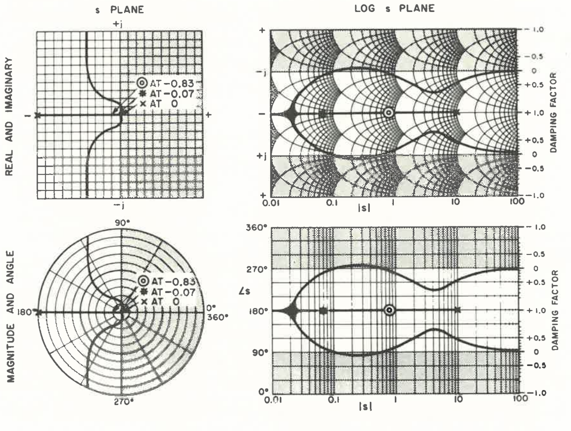


FIGURE I. THE s PLANE VERSUS THE LOG s PLANE

Other coordinate systems may be named by the function of the variable defining them. The coordinate system to be discussed in this paper is the log s plane, shown in the right half of Figure 1. Its rectangular coordinates are $\text{Re}[\log s]$ and $\text{Im}[\log s]$. The same sample root locus plot (to be discussed later) is shown in each of the four coordinate systems.

The transformation of polar graph paper into semilogarithmic is the simpler one to understand. Since $\log s = \log|s| + j\angle s$, the radial distance from the origin in the s plane becomes a logarithmic horizontal scale for $|s|$; the angular position becomes a linear vertical scale for $\angle s$. Although root locus and pole-zero plots in the log s plane at first seem strange to those accustomed to working in the s plane, it soon becomes easy to visualize the transformation from polar to semilog coordinates.

The rectangular (real and imaginary) coordinates of the s plane transform into the curvilinear coordinates shown to their right. To understand this transformation compare it with the polar to semilog transformation. Note that the four arms of the s plane axes-- $+, +j, -, -j$; or $0°, 90°, 180°, 270°$--have become parallel horizonal lines in the log s plane, with logarithmic scales instead of linear. Consider first the upper right quadrant of the s plane. Imagine that the rectangular graduations are lines printed on an elastic membrane. Now hold the $+$ arm stationary and rotate the $+j$ arm clockwise and lift it up so that it is parallel to and above the $+$ arm. Stretch the membrane at the left and compress it at the right to make a logarithmic horizontal scale; the result is the bottom band of the curvilinear log s graph paper. In a like manner, rotate and move each succeeding arm until all four quadrants are stacked up with the $+$ arm repeating itself at the top edge of the graph. The origin of each arm is off the left edge of the graph and the maximum value of each is at the right edge. Reading this presentation is quite simple. If a point falls in the band between $+$ and $+j$, for example, follow one curved coordinate to the $+$ axis for the real value, and the other curved coordinate to the $+j$ axis to read the imaginary value. Since all the axes are parallel and have the same logarithmic scale, their values need only be shown along either the top or bottom edge of the sheet of graph paper.

The only point in the s plane which can never be shown in the log s plane is $s=0$. This is no great disadvantage--it simply requires that the order of the zero or pole at $s=0$ be listed so that s^{n0} can be included in the calculations. The value of K must accompany the pole-zero plot also.

The ratio of vertical and horizontal scales in the log s plane is chosen so that a factor of ϵ (2.718) on the $|s|$ scale is equal to a difference of one radian $(57.3°)$ on the $\angle s$ scale. This makes the transformation conformal, which is necessary when potential analog methods are utilized. The use of the same proportions for graphical work is convenient for comparing with the results from potential analog computations. The conformal transformation has the advantage of maintaining angles of intersections and shapes in any small incremental region. For example, circles transform essentially into circles and squares into squares if they are small enough.

Semi-log and log-log graph papers in these proportions, and curvilinear graph paper for $\mathrm{Re}[s]$ and $\mathrm{Im}[s]$ in the log s plane, are available at a scale of four decades in ten inches for use with the ESIAC® Algebraic Computer. There is also one type of general-purpose semi-logarithmic graph paper which very nearly meets this ratio criterion. Three decades (cycles) in ten inches by ten lines per half-inch, when each line represents two degrees, fits the conformal transformation ratio within 1 percent. Log-log graph paper is available to the same scale, and semi-logarithmic paper is also available with the same logarithmic scale but with twelve lines per inch, providing forty lines to represent twenty decibels in the same space as a decade.

FREQUENCY RESPONSE FROM ZEROS AND POLES

Frequency response methods can be looked upon as a special case of the Laplace transformation. When the variable s in a transfer function is replaced by the complex frequency $j\omega$ or $j2\pi f$, the magnitude and angle of the function represent the gain and phase shift of the system for a sinusoidal input. Plots of log gain (decibels) and phase shift versus log frequency are commonly called Bode plots.

One of the principal advantages of the Bode method is the ease with which it provides sketches of gain and phase for a transfer function. On a logarithmic gain scale or a linear angle scale, a curve for each term in the logarithmic equations (1a) and (1b) or (2a) and (2b) could be plotted separately, then the results added graphically. The Bode method carries this procedure one step farther. When the frequency scale also is logarithmic, the plot for each term approaches a straight line when ω is much smaller or much larger than $|s_i|$. It is therefore very useful to approximate the gain and phase curves by "asymptote plots"[1,4] in which the curves are approximated by straight line segments intersecting at $\omega = s_i$. This straight line approximation is equivalent to always dropping the smaller of the two terms in each factor--for example, the factor $(1 - s/s_i)$ is assumed equal to 1 when s is smaller than s_i, and equal to (s/s_i) when s is larger than s_i. This produces a gain plot with a break in slope at $\omega = s_i$, and a phase plot with a step at this value of ω.

Figure 2 compares the asymptote plots and the actual plots for a factor containing a single zero, and for a pair of factors containing complex conjugate zeros. For poles, turn these plots upside down. Note that the difference in shape between the actual gain or phase curve and its asymptote approximation depends only on the vertical or $\angle s$ distance between them; when zeros or poles are moved to the left or right, their gain and phase curves move left or right the same distance without changing size or shape. The curves in Figure 2 are labeled in terms of the damping factor $\zeta = -\cos \angle s$, instead of $\angle s$ itself.

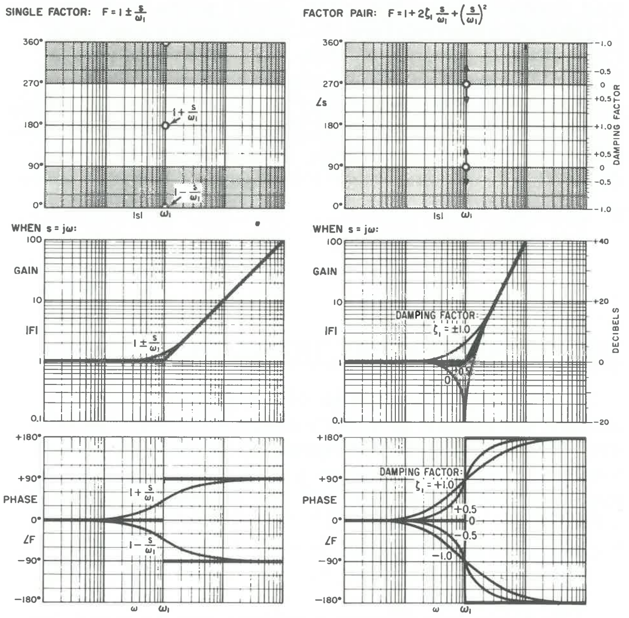


FIGURE 2. FREQUENCY RESPONSE CONSTRUCTION CURVES

When the frequency response plots are compared with the logarithmic pole-zero plot for a typical problem (see Figures 6 and 8), the ease of converting from either one to the other is immediately apparent. The breaks in the straight-line asymptote plots line up with the horizontal zero and pole positions, and the corrections in the vicinity of each break are related to the vertical positions of their zeros or poles by the master curves in Figure 2.

LOGARITHMIC ROOT LOCUS PLOTTING

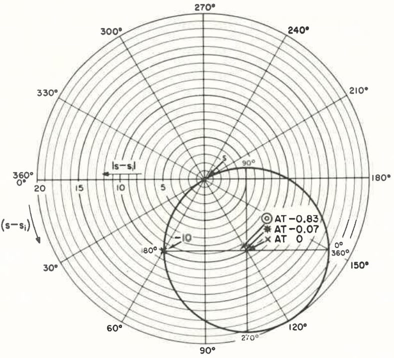
When a function in the form of Equation (1) or (2) is to be evaluated for a value of s not on the $j\omega$ axis, a different method is required. Such calculations are necessary in root locus plotting, and also in calculating residues at the poles of a function to determine transient response.

The root locus method in general has the advantage over frequency response methods that it locates all the poles of a system even though several quadratic pairs may exist. The Evans root locus plot is substantially more time-consuming to construct graphically than the Bode plot, but with the availability of analog calculating aids and the increased accessibility of digital computers, it has begun to compete because of its ability to locate all the poles of a system.

The principal disadvantage of the Evans plot is that its linear scale often makes necessary replotting on a different scale to emphasize a different frequency range. Aids such as the Spirule[5] and the Complex Plane Analyzer[6] have been based on the measurement of $|s-s_i|$ and $\angle(s-s_i)$ as the length and direction of the vector between **s** and s_i in the **s** plane. To evaluate Equations **(1a)** and **(1b)**, the vector lengths or distances are multiplied (or their logarithms added) and their angles added. Unfortunately, this technique is limited to the **s** plane.

In the **s** plane, similar results can also be obtained by placing a transparent polar coordinate graph paper over the pole-zero plot, moving its center to the desired value of **s**, and reading the coordinates of each pole and zero on the transparent overlay. This is illustrated in the top half of Figure 3. A similar procedure can be used for the log **s** plane as shown in the bottom half of the figure. The principal difference between the **s** plane and the log **s** plane in this method is that in the log **s** plane the values read from the transparent overlay yield the magnitude and angle of $(1-s/s_i)$ instead of $(s-s_i)$, corresponding to Equations **(2a)** and **(2b)** instead of **(1a)** and **(1b)**. The application of this technique to root locus plotting in the log **s** plane is described by Nomoto.[7]

FOR CALCULATING $F = s^{n_0} \prod_i (s-s_i)^{n_i}$ IN THE s PLANE



FOR CALCULATING $F = s^{n_0} \prod_i \left(1-\frac{s}{s_i}\right)^{n_i}$ IN THE LOG s PLANE

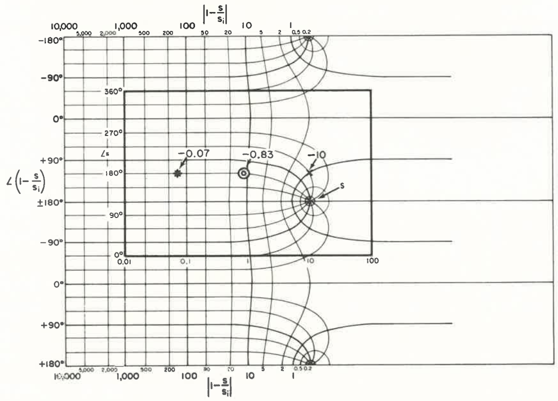


FIGURE 3. GRAPHICAL CALCULATION IN THE s PLANE AND THE LOG s PLANE

The practical difficulty with all of the above methods for root locus plotting is still one of speed, since trial and error solutions are required to determine unknown values of **s**. In these methods the values of $|F|$ and $\angle F$ are not displayed instantaneously, but must be determined by adding separate measurements for each trial value of **s**, which is a tedious procedure. A new method of using potential plane analogs[8, 9] was developed to overcome this difficulty. It uses a pair of electrical voltage distributions analagous to the set of curves in the transparent overlay charts in Figure 3. This principle is used in the ESIAC® Algebraic Computer,[10, 11] which is specifically designed for root locus plotting, frequency response plotting and residue measurement--all in logarithmic coordinates. The plotting center of the ESIAC is shown in Figure 4.

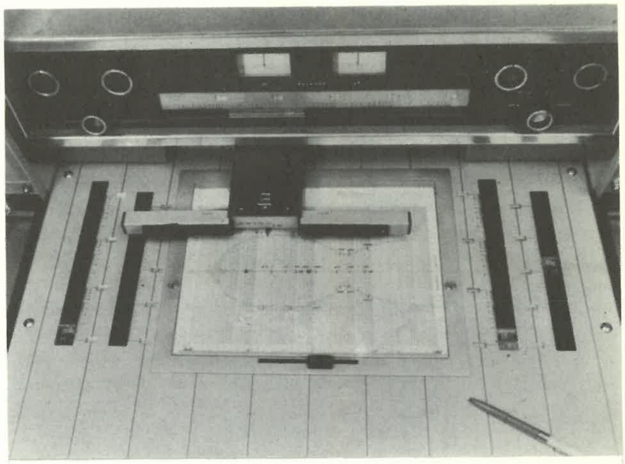


FIGURE 4. CURVE PLOTTING WITH THE ESIAC ALGEBRAIC COMPUTER

The ESIAC accepts equations in the form of **(1a)** and **(1b)** as well as **(2a)** and **(2b)**. It provides continuous indication of values for both the magnitude and angle equations and is equipped for automatic marking of solutions of the equations. Its plotting mechanism can be coupled to draw logarithmic frequency response curves and logarithmic root locus plots. It is not limited to rational functions--zeros and poles need not occur in conjugate pairs, and factors with fractional exponents can be included.

AN ILLUSTRATIVE PROBLEM

The use of the logarithmic complex plane for pole-zero and root locus plots relates this method more closely to the frequency response method and encourages the transfer back and forth between the two methods to make the most effective use of each. The illustrative problem given as a block diagram in Figure 5 will be used to demonstrate how the techniques might be combined.

BLOCK DIAGRAM

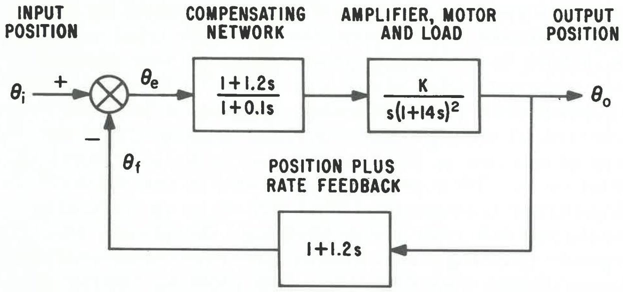


FIGURE 5. AN ILLUSTRATIVE PROBLEM

The open loop function for this system is:

$$\frac{\theta_f}{\theta_e} = K\frac{(1+1.2s)^2}{s(1+14s)^2(1+0.1s)} \qquad (4)$$

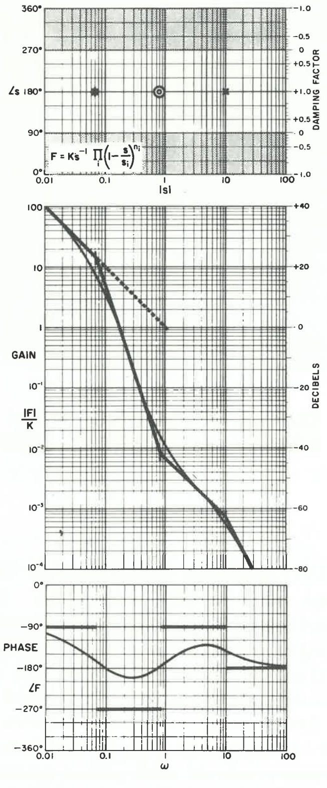


FIGURE 6. OPEN LOOP PLOTS FOR ILLUSTRATIVE PROBLEM

The pole-zero diagram and frequency response curves for this equation are shown in Figure 6. Note how easily the gain and phase asymptote plots can be drawn from inspection of the pole-zero plot. Both the gain and phase plots have their break points directly in line with the poles and zeros above. Each line segment in the gain plot has a slope equal to the algebraic sum of all the zeros and poles (weighted in proportion to their orders) to the left of the segment. The extension of the segment from zero frequency (dotted) passes through a gain of 1 at $\omega=1$. The phase asymptote plot is in the form of steps. Since this transfer function has no poles or zeros in the shaded bands of the pole-zero diagram (is stable and minimum-phase), each step of the phase asymptote is at the multiple of 90° corresponding to the slope of the magnitude plot directly above. The actual gain and phase curves can be constructed from the asymptote plots by using measurements with a pair of dividers on the master curves in Figure 2 or by means of a set of templates constructed to the shape of actual curves for different damping factors. With an ESIAC, the actual curves are plotted directly from the pole-zero plot.

To analyze the system by the frequency response method, the gain and phase curves in Figure 6 are studied to select a suitable value of K. Note that the system will be only conditionally stable, since the phase shift exceeds 180° for ω values between 0.1 and 0.7. The $|F|/K$ values at these frequencies are 3.3 and 0.02, therefore for Kvalues between 0.3 and 50 the system will be unstable; at higher values of K the system will be stable. The maximum phase margin is reached at at $\omega \approx 4$, corresponding to $|F|/K \approx 0.0016$, or $K \approx 600$.

The alternate method of analyzing the problem is the root locus method. Figure 7 shows

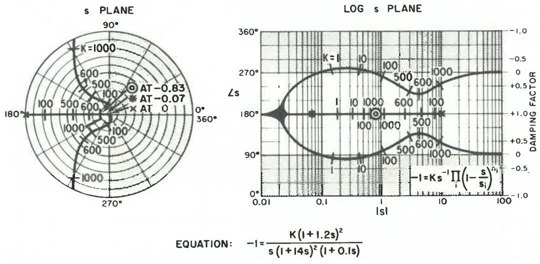


FIGURE 7. ROOT LOCUS PLOTS FOR ILLUSTRATIVE PROBLEM

the root locus plot corresponding to Figure 6. Both the log **s** plane and the **s** plane root locus plots are included to show their comparison. In the **s** plane, such a root locus plot is usually constructed using a Spirule; in the log **s** plane, it is constructed by the graphical overlay method of Figure 2 or by using and ESIAC, shown with this example in Figure 4.

Note in Figure 7 that the root locus plot in the s plane suffers from limited range--only one decade is shown with reasonable detail. With the scale chosen, the region of maximum damping is well shown, but very little information is given near the origin, where the system is unstable. If the scale were expanded for more detail in this region, the optimum operating region would be off the paper. In the log s plane, on the other hand, the root locus plot is shown over the same four-decade range as the frequency response curves.

In the logarithmic root locus plot, it is again seen that the system is unstable between $|s|$ values of 0.1 and 0.7, where K is between 0.3 and 50. A maximum damping factor of 0.7 for the oscillatory roots is found at $\omega \approx 4$, where $K \approx 600$. Thus, both the open-loop frequency response in Figure 6 and the logarithmic root locus plot in Figure 7 give the same basic information.

The advantage of the root locus plot is that its root locations are the poles of the closed-loop system, from which the closed-loop frequency response or transient response can be directly determined. Note that the closed-loop function for the system of Figure 5 can be written as:

$$\frac{\theta_0}{\theta_i} = \frac{K(1+1.2s)}{K(1+1.2s)^2 + s(1+14s)^2(1+0.1s)} \qquad (5)$$

To factor the denominator, set it equal to zero:

$$K(1+1.2s)^2 + s(1+14s)^2(1+0.1s) = 0 \qquad (6)$$

This is the same as setting the open-loop function in Equation (4) equal to -1:

$$K\frac{(1+1.2s)^2}{s(1+14s)^2(1+0.1s)} = -1 \qquad (7)$$

This is the equation which is solved by means of the root locus plot in Figure 7 for all values of K.

Designating the roots obtained in Figure 7 as s_1, s_2, s_3 and s_4, the denominator of Equation (5) can be replaced by:

$$K\left(1-\frac{s}{s_1}\right)\left(1-\frac{s}{s_2}\right)\left(1-\frac{s}{s_3}\right)\left(1-\frac{s}{s_4}\right)$$

Substituting this expression in Equation (5) yields the overall closed-loop transfer function in factored form:

$$\frac{\theta_0}{\theta_i} = \frac{(1+1.2s)}{\left(1-\frac{s}{s_1}\right)\left(1-\frac{s}{s_2}\right)\left(1-\frac{s}{s_3}\right)\left(1-\frac{s}{s_4}\right)} \qquad (8)$$

This equation is shown as a logarithmic pole-zero plot at the top of Figure 8; the pole locations s_1, s_2, s_3 and s_4 in Equation (8) and Figure 8 are the root locations in Figure 7 for $K=600$. Again, as in Figure 6, the frequency response curves can be drawn either by sketching the asymptotes and applying corrections, or by drawing them directly on an ESIAC.

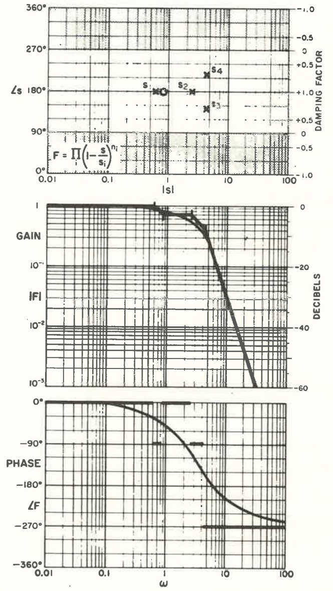


FIGURE 8. CLOSED LOOP PLOTS FOR ILLUSTRATIVE PROBLEM (K = 600 IN FIGURE 7)

The closed-loop pole-zero plot can also be used to determine the response of the system to a transient input. To calculate a transient response, express the transfer function in the $(s-s_i)$ factor form and multiply it by the transform of the specified input.

In the $(s-s_i)$ factor form, the transfer function for this illustration is:

$$\frac{\theta_0}{\theta_i} = \frac{36.7(s+0.83)}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \qquad (9)$$

Note that the 'K' multipliers in both Equation (8) and Equation (9) were obtained by inspection of Equation (5), in which the coefficient of the lowest power of s is $K/K=1$ and the coefficient of the highest power of s is $K \times 1.2/(14 \times 14 \times 0.1) = 36.7$ for $K=600$. For a unit step input $\theta = 1/s$, for example, the output of this system will be:

$$\theta_0 = \frac{36.7(s+0.83)}{s(s-s_1)(s-s_2)(s-s_3)(s-s_4)} \qquad (10)$$

which can be rewritten in the form:

$$\theta_o = \frac{A_0}{s} + \frac{A_1}{s-s_1} + \frac{A_2}{s-s_2} + \frac{A_3}{s-s_3} + \frac{A_4}{s-s_4} \qquad (11)$$

Each A in Equation (11) is equal to the value of Equation (10) when s is set equal to the corresponding pole and that factor is removed from the equation; this value is called the residue at that pole.* It is particularly easy to determine residues at first-order poles with an ESIAC--when s is set equal to a pole, the factor containing that pole is automatically canceled and the magnitude and angle of the residue read directly on the ESIAC dials.

The inverse transform of Equation (11) is:

$$\theta_o(t) = A_0 + A_1 e^{s_1 t} + A_2 e^{s_2 t} + A_3 e^{s_3 t} + A_4 e^{s_4 t} \qquad (12)$$

In transform equations such as (11), real pole values have real residue values, and complex pole pairs have complex conjugate residue values. Terms containing complex conjugate pairs can therefore be combined by the formula:

$$Ae^{(\sigma+j\omega)t} + A^{*}e^{(\sigma-j\omega)t} = 2|A|e^{\sigma t}\cos\left(\angle A + \frac{360^\circ}{2\pi}\omega t\right) \qquad (13)$$

Note that $|A|$ and $\angle A$ are the magnitude and angle of the residue at $s=\sigma+j\omega$; the residue A^{*} at $s=\sigma-j\omega$ need not be calculated as its effect is already included in the equation.

The residue values and inverse transform expressions for this example are listed in Table I. The inverse transform for each term can then be sketched and the results added graphically as shown in Figure 9; this results in the actual output waveform as it would be seen on an oscilloscope.

POLE s_i	RESIDUE A_i	INVERSE TRANSFORM $A_i e^{s_i t}$
$s_0 = 0$	$A_0 = 1\angle 0^\circ$	$+1$
$s_1 = 0.63\angle 180^\circ = -0.63$	$A_1 = 0.40\angle 180^\circ$	$-0.40e^{-0.63t}$
$s_2 = 2.5\angle 180^\circ = -2.5$	$A_2 = 1.38\angle 180^\circ$	$-1.38e^{-2.5t}$
$s_3 = 4.4\angle +137^\circ = -3.2+j3.0$	$A_3 = 0.44\angle +27^\circ$	$+0.88e^{-3.2t}\cos\left(27^\circ + \frac{360^\circ}{2\pi}3.0t\right)$
$s_4 = 4.4\angle -137^\circ = -3.2-j3.0$	$A_4 = 0.44\angle -27^\circ$	

TABLE I. RESIDUE VALUES AND INVERSE TRANSFORMS

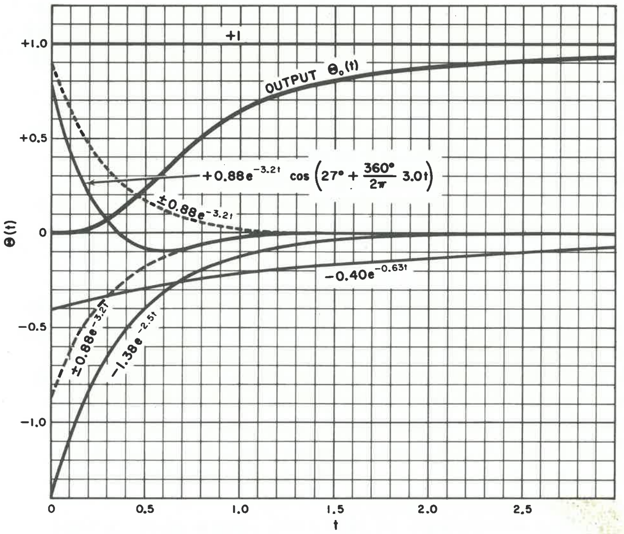


FIGURE 9. GRAPHICAL CONSTRUCTION OF TRANSIENT RESPONSE

SUMMARY

This paper has discussed the characteristics of the logarithmic complex plane, its use in pole-zero and root locus plotting, and its relationship to logarithmic frequency response methods. The construction of logarithmic frequency response plots has been reviewed and the construction of logarithmic root locus plots described using graphical techniques and using the ESIAC Algebraic Analog Computer. The mixing of frequency response and root locus methods, the sketching or visualizing of frequency response curves from logarithmic pole-zero plots and vice versa, and the construction of transient response curves from pole-zero plot calculations have been demonstrated by an illustrative problem.

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* This applies to first-power denominator factors only. For finding coefficients at higher order poles, see reference 12. For a general discussion of partial fraction expansions, refer to any textbook on complex algebra.

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